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Regularity of the Attractor for Schrödinger Equation

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Abstract—We prove that the global attractor for a weakly damped nonlinear Schrödinger equation is smooth, i.e., it is made of smooth functions when the forcing term is smooth. Our study relies on a new approach that works for dispersive equations that are weakly dissipative, i.e., for equations for which the damping is on the low-order term.

Keywords—Nonlinear Schrödinger equations, Weak damping, Global attractor.

1. INTRODUCTION

In this work, we are interested in some issues concerning the long-time behavior of solutions of a weakly damped nonlinear Schrödinger equation that reads

$$u_t + \alpha u + i u_{xx} - iu + i|u|^2 u = f, \tag{1.1}$$

where the unknown u maps $\mathbb{R}_+$ into the space of complex valued functions

$$H^1 = \left\{u : [0,1] \to \mathbb{C}; u \text{ and } u_x \text{ belong to } L^2(0,1), \text{ and } x \mapsto u(x) \text{ is 1-periodic}\right\};$$

here the damping $\alpha > 0$ and the force f, which is assumed to be a smooth periodic function, are given.

It is well known that the Cauchy problem related to (1.1) and to the initial condition

$$u(0) = u_0, \qquad \text{in } H^1 \tag{1.2}$$

is well posed. This allows us to define a semigroup $S(t)$ acting on H^1 by setting

$$S(t)u_0 = u(t), \tag{1.3}$$

where $u(t)$ solves (1.1),(1.2).

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Actually, this equation is dissipative, i.e., all solutions enter an absorbing ball in H^1 after a transient time. Moreover, the semigroup $(S(t))_{t\geq 0}$ possesses a global attractor $\mathcal{A}$ in H^1. Let us overview the history of this result. In [1], the author proved the existence of a *weak* global attractor $\mathcal{A}$ for the semigroup, i.e., the existence of a bounded weakly closed subset of H^1, invariant for the flow of the solutions, and attracting all the trajectories (in H^1 endowed with its weak topology) when t goes to $+\infty$. This result was improved in [2], where the author establishes that this weak attractor $\mathcal{A}$ is, in fact, a global attractor in H^1, i.e., attraction holds in the usual strong topology (see e.g., [3]).

In this note, we aim to present the following new result: let k be any nonnegative integer. Then, the global attractor $\mathcal{A}$ is included and bounded in H^{k+2}, provided the forcing term f belongs to H^k. This describes an *asymptotical smoothing effect* for equation (1.1). This equation is not regularizing, $S(t)u_0$ belongs to H^1 when u_0 is in H^1, *but* it features a smoothing effect when $t \to +\infty$, since all trajectories converge towards a set which is made of smooth functions.

Moreover, our approach provides us with a new simpler proof of the existence of this global attractor. All these results and their proofs will appear in [4]. The method also extends to weakly damped nonlinear Schrödinger equations on the whole line $\mathbb{R}$ and to higher space dimensions [5].

2. A NEW APPROACH

Let $u(t)$ be the solution of (1.1),(1.2). We expand $u(t)$ into its Fourier series that reads

$$u(t) = \sum_{k\in\mathbb{Z}} u_k(t) e^{2i\pi kx}. \tag{2.1}$$

On the one hand, for a given level N, the low frequency part of u,

$$y(t) = \sum_{|k|\leq N} u_k(t) e^{2i\pi kx}, \tag{2.2}$$

is a smooth function with respect to the x variable (an analytical one). Therefore, the regularity of u with respect to x depends on the regularity of its high frequency part, that is

$$z(t) = \sum_{|k|> N} u_k(t) e^{2i\pi kx}. \tag{2.3}$$

On the other hand, z is solution of the nonautonomous partial differential equation

$$z_t + \alpha z + i z_{xx} - iz + iQ\left(|y+z|^2(y+z)\right) = Qf, \tag{2.4}$$

with initial condition

$$z = Qu_0 = z_0, \tag{2.5}$$

where Q denotes the orthogonal projector onto the high-frequency space

$$QH^1 = \overline{\operatorname{span}}\left(e^{2i\pi kx}; |k| > N\right). \tag{2.6}$$

We now introduce $Z : \mathbb{R}_+ \to QH^1$, which is the solution of

$$\begin{aligned} Z_t + \alpha Z + iZ_{xx} - iZ + iQ\left(|y+Z|^2(y+Z)\right) &= Qf, \\ Z(0) &= 0. \end{aligned} \tag{2.7}$$

We can state the following theorem.

THEOREM 2.1. *Let us assume (for convenience) that $u(t)$ belongs for $t \geq 0$ in the absorbing ball in H^1. Let $y(t) = (Id - Q)u(t)$ be as above. Let us assume that f belongs to H^k, $k \geq 0$. Let N be fixed (large enough). Then, there exists a (unique) solution Z of (2.7), which is continuous and bounded from $\mathbb{R}^+$ into QH^{k+2}. Moreover,*

$$\overline{\lim_{t\to+\infty}} ||Z(t) - z(t)||_{H^1} = 0. \tag{2.8}$$

The (complete) proof of this theorem will appear in [4].

3. APPLICATION TO THE ATTRACTOR

The previous theorem allows us to follow the guidelines of Theorem I.1.1 in [3], since the solution $u(t)$ of (1.1),(1.2) splits into a regular part $y(t) + Z(t)$, and a small one $z(t) - Z(t)$ which converges to 0 when $t \to +\infty$. Hence, we have the following theorem.

THEOREM 3.1. *The semigroup $(S(t))_{t\geq 0}$ defined by (1.1),(1.2) possesses a compact global attractor $\mathcal{A}$ in H^1. Moreover, $\mathcal{A} \subset H^{k+2}$ when $f \in H^k$, $k \geq 0$.*

We conclude this note by stating two other applications. First, it is well known that solving (1.1) with initial condition

$$u_0 \in H^m, \tag{3.1}$$

where m is an integer larger than 1, leads to a well-posed problem; this allows us to define a sequence of semigroups, still denoted $S(t)$, that acts on H^m (here we assume that $f \in \cap_{k\geq 0} H^k$). Moreover, each of these semigroups possesses a global attractor $\mathcal{A}_m$ in H^m. Indeed, as a consequence of Theorem 3.1, we have the following proposition.

PROPOSITION 3.2.

$$\mathcal{A}_m = \mathcal{A}_{m-1} = \cdots = \mathcal{A}.$$

The last application is the fact that if N is large enough, a trajectory $u(t)$ that belongs to $\mathcal{A}$ is completely determined by its projection onto the low-frequency space. This result has been observed in [6] for the two-dimensional Navier-Stokes equations (see also [7]). Here we have the following proposition.

PROPOSITION 3.3. *Let $u^1(t)$, $u^2(t)$, $\forall\, t \in \mathbb{R}$ be two complete orbits that belong to $\mathcal{A}$. Then,*

$$(Id - Q)u^1(t) = (Id - Q)u^2(t), \qquad \forall\, t \in \mathbb{R} \Rightarrow u^1(t) = u^2(t), \qquad \forall\, t \in \mathbb{R}. \tag{3.2}$$

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